HEAT TRANSFER IN A CYLINDRICAL LAYER OF AN ABSORBING MEDIUM BOUNDED BY NONBLACK SURFACES

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UDC 536.3

The temperature field and the radiative–conductive heat flow in a cylindrical layer of a weakly-absorbing medium are computed.

We consider a cylindrical layer bounded by surfaces with reflection factors R_1 and R_2 and filled by a medium with absorption coefficient α_ν, refractive index n_ν, and molecular thermal conductivity λ_M.

In solving the problem of the temperature field in the layer with given temperature at its boundaries, we make the following assumptions.

1. The thermal flows due to molecular thermal conductivity and radiative heat transfer can be combined additively.

2. The absorption and natural radiation in an elementary volume of the medium are linked by Kirchhoff's law:

$$e_\nu = \alpha_\nu n_\nu^2 \varepsilon_\nu,$$

where e_ν is the volume coefficient of radiation of the medium.

3. The quantities α_ν, n_ν, and λ_M are independent of the temperature within the limits of the temperature difference between the walls.

4. The surfaces bounding the layer reflect uniformly in all directions.

With these assumptions we solve the equation

$$\operatorname{div}(Q_M + Q_R) = 0 \tag{1}$$

with the boundary conditions

$$\begin{aligned} T(r_1) &= T_1, \\ T(r_2) &= T_2, \end{aligned} \tag{2}$$

where Q_M is defined by Fourier's law. To find Q_R we solve the radiation intensity transport equation

$$\frac{dI_\nu}{dS} = -\alpha_\nu I_\nu + \alpha_\nu n_\nu^2 \varepsilon_\nu. \tag{3}$$

For each fixed direction S, defined by the two angles θ and γ, the derivatives of the intensity along the radius and along the fixed direction S are linked by the equation

$$\frac{dI}{dS} = \frac{dI}{dr}\frac{dr}{dS}, \tag{4}$$

where

$$\frac{dr}{dS} = \mp \frac{\left|\sqrt{r^2 - r_2^2 \sin^2\theta}\right|}{r}\cos\gamma, \tag{5}$$

Institute of Crystallography, Academy of Sciences of the USSR, Moscow. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 18, No. 1, pp. 31-38, January, 1970. Original article submitted January 27, 1969.

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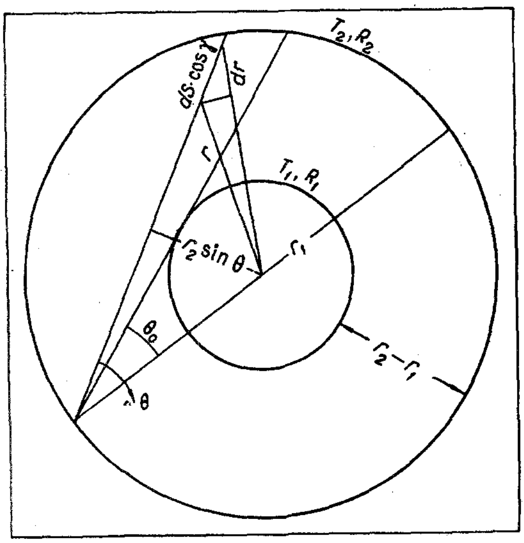

Fig. 1. Normal section of the cylindrical layer.

and θ is the angle between the projection of S on a normal section of the cylinder and the greatest diameter of the cylinder; γ is the angle which the vector **S** makes with the plane of the normal section in the plane θ = const (see Fig. 1). The derivative dr/dS changes sign at the point $r = r_2 \sin\theta$. From (4) and (5), Eq. (3) takes the form

$$\frac{dI_\nu^+}{dr}\,\frac{\left|\sqrt{r^2-r_2^2\sin^2\theta}\right|}{r}\cos\gamma = -\alpha I_\nu^+ + \alpha_\nu n_\nu^2\varepsilon_\nu{}^*, \tag{6}$$

$$\frac{dI_\nu^-}{dr}\,\frac{\left|\sqrt{r^2-r_2^2\sin^2\theta}\right|}{r}\cos\gamma = \alpha I_\nu^- - \alpha_\nu n_\nu^2\varepsilon_\nu. \tag{7}$$

The sign "+" corresponds to the direction in which r increases, the sign "−" to that in which it decreases.

We also have to distinguish the directions which intersect the internal surface of the layer ($0 \le \theta \le \theta_0$, $\sin\theta_0 = r_1/r_2$) and those which do not ($\theta_0 \le \theta \le \pi/2$). The boundary conditions can be written as

$$\begin{gathered} I^+(r_1,\ \theta,\ \gamma) = \varepsilon(T_1)\,n^2(1-R_1) + R_1 I^-(r_1,\ \theta,\ \gamma),\\ I^-(r_2,\ \theta,\ \gamma) = \varepsilon(T_2)\,n^2(1-R_2) + R_2 I^+(r_2,\ \theta,\ \gamma),\\ 0 \leqslant \theta \leqslant \theta_0,\\ I^-(r_2,\ \theta,\ \gamma) = \varepsilon(T_2)\,n^2(1-R_2) + R_2 I^+(r_2,\ \theta,\ \gamma),\\ \theta_0 \leqslant \theta \leqslant \frac{\pi}{2}. \end{gathered} \tag{8}$$

By solving (6) and (7) with the boundary conditions (8), we can obtain the difference $I^+ - I^-$. We know that in the general case the radiative component of the heat flow can be expressed as

$$Q_R = \iint (I^+ - I^-)\cos(r,\ s)\,d\omega d\nu, \tag{9}$$

where $d\omega$ is the element of solid angle.

It can be shown that for a cylindrical layer

$$\cos(r,\ S)\,d\omega = \frac{r_2}{r}\cos\theta\cos^2\gamma\,d\theta d\gamma. \tag{10}$$

Substituting $I^+ - I^-$ and (10) in (9), we obtain an expression for the radiative heat flow

$$\begin{aligned} Q_R = \frac{4r_2}{r}\int_0^\infty n^2 \int_0^{\pi/2} \Bigg\{ \int_0^{r_1/r_2} \beta \Bigg[&-\int_{r_1}^{r} \exp\left(-\alpha\frac{x-x'}{\cos\gamma}\right)\frac{\partial\varepsilon}{\partial r'}dr' \\ &+ \int_{r_2}^{r} \exp\left(-\alpha\frac{x'-x}{\cos\gamma}\right)\frac{\partial\varepsilon}{\partial r'}dr' + R_1\int_{r_1}^{r_2} \exp\left(-\alpha\frac{x+x'-r_1'}{\cos\gamma}\right)\frac{\partial\varepsilon}{\partial r'}dr' \\ &+ R_2\int_{r_1}^{r_2} \exp\left(-\alpha\frac{2r_2'-x-x'}{\cos\gamma}\right)\frac{\partial\varepsilon}{\partial r'}dr' - R_1R_2\int_{r_1}^{r} \exp\left(-\alpha\frac{2r_2'+x-x'-2r_1'}{\cos\gamma}\right)\frac{\partial\varepsilon}{\partial r'}dr'\Bigg]dt \\ &+ \int_{r_1/r_2}^{r/r_2} \chi\Bigg[-\int_{r_2t}^{r} \exp\left(-\alpha\frac{x-x'}{\cos\gamma}\right)\frac{\partial\varepsilon}{\partial r'}dr' - \int_{r}^{r_2} \exp\left(-\alpha\frac{x'-x}{\cos\gamma}\right)\frac{\partial\varepsilon}{\partial r'}dr' \\ &+ \int_{r_2t}^{r_2} \exp\left(-\alpha\frac{x+x'}{\cos\gamma}\right)\frac{\partial\varepsilon}{\partial r'}dr' - R_2\int_{r_2t}^{r} \exp\left(-\alpha\frac{2r_2'+x'-x}{\cos\gamma}\right) \\ &\times\frac{\partial\varepsilon}{\partial r'}dr' - R_2\int_{r}^{r_2} \exp\left(-\alpha\frac{2r_2'+x-x'}{\cos\gamma}\right)\frac{\partial\varepsilon}{\partial r'}dr' + R_2\int_{r_2t}^{r_2} \exp\left(-\alpha\frac{2r_2'-x-x'}{\cos\gamma}\right)\frac{\partial\varepsilon}{\partial r'}dr'\Bigg]dt\Bigg\}\cos^2\gamma d\gamma d\nu, \end{aligned} \tag{11}$$

* In what follows the sign of the modulus and the subscript ν will be omitted.

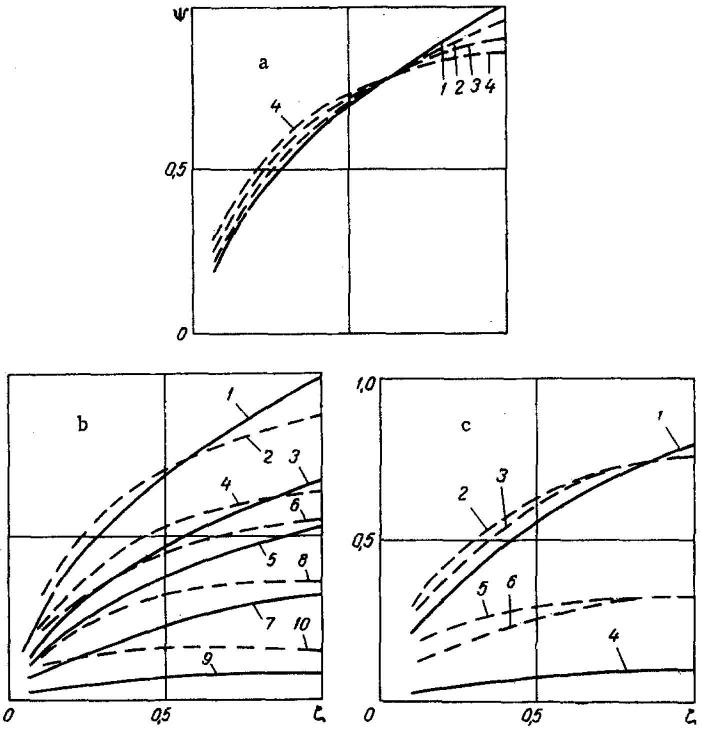

Fig. 2. The nondimensional radiative heat flow Ψ as a function of ζ for $R_1 = R_2 = 0 - a$; for $R_1 = R_2$ – b; and for $R_1 \neq R_2$ – c: a) 1 – $\alpha L = 0$; 2 – 0.1; 3 – 0.2; 4 – 0.3; b) 1 – $\alpha L = 0$; $R_1 = R_2 = 0$; 2 – 0.2 and 0; 3 – 0 and 0.2; 4 – 0.2 and 0.2; 5 – 0 and 0.3; 6 – 0.2 and 0.3; 7 – 0 and 0.5; 8 – 0.2 and 0.5; 9 – 0 and 0.9; 10 – 0.2 and 0.9; c) 1 – $\alpha L = 0$; $R_1 = 0$, $R_2 = 0.2$; 2 – 0.2, 0, 0.2; 3 – 0.2, 0.2, 0; 4 – 0, 0, 0.9; 5 – 0.2, 0, 0.9; 6 – 0.2, 0.9, 0.

where

$$x = \sqrt{r^2 - r_2^2 t^2};\quad x' = \sqrt{(r')^2 - r_2^2 t^2};\quad r_1' = \sqrt{r_1^2 - r_2^2 t};$$

$$r_2' = \sqrt{r_2^2 - r_2^2 t};\ t = \sin\theta;$$

$$\beta = \frac{1}{1 - R_1 R_2 \exp\left(-2\alpha \frac{r_2' - r_1'}{\cos\gamma}\right)};\quad \chi = \frac{1}{1 - R_2 \exp\left(-2\alpha \frac{r_2'}{\cos\gamma}\right)}.$$

Equation (1), which has the following form for a cylindrical layer:

$$\lambda_{\text{M}}\left(\frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr}\right) = \frac{1}{r}\frac{d(rQ_R)}{dr}, \tag{12}$$

where Q_R is defined by (11), is in general a nonlinear second order integro-differential equation.

In what follows we consider a linear approximation of this equation, i.e., the case when

$$\frac{\Delta T}{T} \ll 1, \tag{13}$$

and we retain only the linear term in the expansion of Planck's function:

$$\frac{\partial\varepsilon}{\partial r'} = \frac{\partial\varepsilon}{\partial T}\frac{dT}{dr'}. \tag{14}$$

Making the substitution

$$\frac{dT}{dr} = \varphi(r)\,u(r), \tag{15}$$

where

$$\varphi(r)=\frac{\Delta T}{r\ln\frac{r_2}{r_1}} \tag{16}$$

is the expression for the temperature gradients in the layer when $Q_R = 0$ and $u(r)$ is a function to be defined, we solve (12) by the method of successive approximations, taking $u(r) = 1$ as the zero order approximation.

We can only obtain $u(r)$ in analytic form for certain special cases.

The greatest interest is in the solution of the case when αL is a small parameter ($L = r_2-r_1$), in terms of which the integrals in (11) depending on αL are expanded. Having obtained $u(r)$ in analytic form and then, using (15) and (16), and the boundary conditions (2), after appropriate transformations, we obtain an expression for the temperature distribution and the heat flow in the first approximation

$$T=T_1+\Delta T\left(1-\frac{\ln\eta}{\ln\zeta}\right)+\frac{\Delta TR^2}{\lambda}\int_0^\infty\frac{\partial\varepsilon}{\partial T}\alpha n^2$$
$$\times\left\{\left[f_1(\eta,\zeta)+\left(\frac{\ln\eta}{\ln\zeta}-1\right)f_1(1,\zeta)\right]+D\left[f_2(\eta,\zeta)+\left(\frac{\ln\eta}{\ln\zeta}-1\right)f_2(1,\zeta)\right]\right\}dv, \tag{17}$$

where

$$f_1(\eta,\zeta)=\eta^2\arcsin\frac{\zeta}{\eta}+3\zeta\sqrt{\eta^2-\zeta^2}-2\zeta^2\arccos\frac{\zeta}{\eta}-\frac{\pi(\eta^2-\zeta^2)}{\ln\zeta}-\frac{\ln\eta}{\ln\zeta}\pi\left(\eta^2-\frac{\zeta^2}{2}\right);$$

$$f_2(\eta,\zeta)=3\zeta\sqrt{\eta^2-\zeta^2}+2\eta^2\arcsin\frac{\zeta}{\eta}$$
$$-2\zeta^2\arccos\frac{\zeta}{\eta}+\eta^2\arccos\frac{\zeta}{\eta}-\frac{\pi}{2}(\eta^2+\zeta^2);\quad \eta=\frac{r}{r_2};\quad \zeta=\frac{r_1}{r_2};$$

$$Q=-\varphi(r)\left\{\lambda_M+\pi L\int_0^\infty\frac{\partial\varepsilon}{\partial T}\,n^2\Psi\,(R_1,\,R_2,\,\alpha L,\,\zeta)\,dv\right\}, \tag{18}$$

and

$$\Psi=F_1(R_1,R_2)\Phi_1(\zeta)-\frac{\alpha L}{\pi}\sum_{n=2}^{5}F_n(R_1,R_2)\Phi_n(\zeta);$$

$$\Phi_1(\zeta)=\frac{\zeta\ln\zeta}{1-\zeta};\quad \Phi_2(\zeta)=\frac{1}{(1-\zeta)^2}\left(6\zeta\sqrt{1-\zeta^2}+\pi\zeta^2+3\arcsin\zeta\right.$$
$$\left.-4\zeta^2\arccos\zeta+\arccos\zeta+\frac{\pi(1-\zeta^2)}{\ln\zeta}-\frac{\pi}{2}\right);\quad \Phi_3(\zeta)=\frac{\ln\zeta}{(1-\zeta)^2}$$
$$\times(2\zeta\sqrt{1-\zeta^2}+2\arcsin\zeta);\ \Phi_4(\zeta)=\frac{\ln\zeta}{(1-\zeta)^2}\pi\zeta^2;\quad \Phi_5(\zeta)=\frac{1}{(1-\zeta)^2}$$
$$\times(6\zeta\sqrt{1-\zeta^2}+4\arcsin\zeta-\pi\zeta^2-4\zeta^2\arccos\zeta+2\arccos\zeta-\pi);$$
$$F_1=\frac{-1+R_1+R_2-R_1R_2}{1-R_1R_2};\quad F_2=1;\quad F_3=\frac{2R_2(1-R_1)^2}{(1-R_1R_2)^2};$$
$$F_4=-\frac{2R_1(1-R_2)^2}{(1-R_1R_2)^2};\quad F_5=D=\frac{R_2-R_1}{1-R_1R_2}.$$

It is easy to verify that (17) and (18) are correct, since they must tend to the corresponding expressions for a flat layer as $\zeta\to 1$.

Putting

$$r_2=r_1+L,\quad r=r_1+y \tag{19}$$

and making the expansions in series

$$\ln\frac{\eta}{\zeta} = \ln\frac{r}{r_1} = \ln\left(1+\frac{y}{r_1}\right) \cong \frac{y}{r_1} - \frac{y^2}{2r_1^2} + \frac{y^3}{3r_1^3},$$
$$\arcsin\zeta = \arcsin\frac{r_1}{r_2} = \arccos\frac{\sqrt{r_2^2-r_1^2}}{r_2} \cong \frac{\pi}{2} - \frac{\sqrt{r_2^2-r_1^2}}{r_2}, \tag{20}$$

and then, having substituted (20) in (17) and (18), and finding the limit as $r_1/r_2 \to 1$, $y/r_1 \to 0$, $L/r_1 \to 0$, we obtain the corresponding expression for the flat layer [1].

By considering (17) and (18) we see that the temperature distribution in a cylindrical layer with weak absorption depends on the optical properties of the walls only when they have different reflection factors.

In considering the expression for the heat flow we draw attention to the fact that when there is conductive and radiative transport the heat flow is proportional not to the real gradients in the layer, but to $\varphi(r)$, i.e., the influence of radiative transport on the heat flow is equivalent to a change in the coefficient of thermal conductivity of the system. The coefficient of proportionality between Q and $\varphi(r)$ can be treated as the effective thermal conductivity of the layer. It is significant that the effective thermal conductivity is not a constant of the medium, since it depends not only on its thermal and optical properties and the optical properties of the surfaces, but also on the dimensions (L) and the configuration of the system (the parameter ξ).

If α_ν is replaced by the Rosseland mean and we neglect the dependence of R_1, R_2 on ν, then Ψ becomes the nondimensional radiative heat flow

$$\Psi = \frac{Q - Q_M}{\varphi(r)\,\pi L \int\limits_0^\infty \frac{\partial\varepsilon}{\partial T}\, n^2 d\nu}. \tag{21}$$

We consider Ψ as a function of ξ for three different cases.

1. The layer is bounded by absolutely black walls.

On Fig. 2a, Ψ is shown as a function of ξ for various values of αL when $R_1 = R_2 = 0$ (dotted curves). The first term in Ψ, which is independent of αL, describes the change in the heat flow due to heat transfer between the walls. The function $\Phi_1(\xi)$ is shown in Fig. 2a, by a continuous line. As the area of the internal surface tends to zero, $\Phi_1(\xi)$ also tends to zero. The second term, which is a function of αL, defines the change in the heat flow due to absorbing and radiating media.

For a flat layer ($\xi = 1, 0$), when the radiation intensity at the walls is large, absorption dominates natural radiation and the presence of an absorbing medium leads to a reduction in the heat flow proportional to αL.

In the case of a "heated filament" ($\xi \to 0$) the heat flows due to heat transfer between the walls are small, as a result of which natural radiation dominates absorption. For $\xi - \xi_0 \cong 0.6$ the heat flow is independent of αL. The point ξ_0 is the same for all αL only for weak absorption. As the absorption increases, the point ξ_0 is displaced to the left when the optical thickness of the layer increases and the curves gradually degenerate into straight lines parallel to the axis of abscissae, since the effective thermal conductivity need not depend on the configuration of the layer for strong absorption.

2. The layer is bounded by walls with the same reflective capabilities ($R_1 = R_2 \neq 0$).

On Fig. 2b, Ψ is shown as a function of ξ for $\alpha L = 0.2$ and various values of R (R = 0, 0.2, 0.3, 0.5, and 0.9). As the reflective capabilities of the walls increase and heat transfer between them decreases, the role of natural radiation, of course, increases. The point ξ_0 moves to the right. When R = 0.29, $\xi_0 = 1.0$. When R increases further, radiation dominates absorption throughout the whole region in which ξ varies.

3. The layer is bounded by walls of different reflective capabilities ($R_1 \neq R_2 \neq 0$).

On Fig. 2c, Ψ is shown as a function of ζ for the cases: $D \cong +0.2$ ($R_1 = 0$, $R_2 = 0.2$); $D = -0.2$ ($R_1 = 0.2$, $R_2 = 0$) (curves 1, 2, 3); $D \cong +0.9$ ($R_1 = 0$, $R_2 = 0.9$); $D = -0.9$ ($R_1 = 0.9$, $R_2 = 0$) (curves 4, 5, 6). The sign of D has a marked effect on the heat flow. An exception is the region of ζ near to 1, where the curves corresponding to the same value of D, but with opposite signs, merge together. Here the optical properties of the surface, which has a large area for unit length, have a stronger influence. If, for example, the external surface of the layer is "blacker" than the internal surface, the contribution of the natural radiation is reduced (the dotted curves 4 and 6).

It must also be noted that as ΔR increases there is a considerable increase in the heat flow due to natural radiation although the heat flow due to heat transfer between the walls remains approximately the same. This is seen from a comparison of the curves $R_1 = R_2 = 0.9$ (Fig. 2b) and $R_1 = 0$, $R_2 = 0.9$ (Fig. 2c).

NOTATION

α_ν	is the absorption coefficient of the medium;
n_ν	is the refractive index of the medium;
ε_ν	is Planck's function;
R_1, R_2	are the reflection factors of the internal and external cylinders respectively;
Q_M	is the conductive heat flow;
Q_R	is the radiative heat flow;
r	is the radius;
r'	is the integration variable;
r_1, r_2	are the radii of the internal and external cylinders respectively;
$L = r_1 - r_2$	is the thickness of the layer;
$\mathbf{S}$	is the radiation direction vector;
θ, γ	are angles corresponding to the direction S;
$I(r, \theta, \gamma)$	is the radiation intensity;
ω	is the solid angle;
T	is the temperature;
T_1, T_2	are the temperatures of the internal and external cylinders respectively;
$\Delta T = T_2 - T_1$	is the temperature difference;
Ψ	is the nondimensional radiative heat flow;
$\eta = r/r_2$	is a nondimensional coordinate;
$\zeta = r_1/r_2$	is the nondimensional configuration parameter;
λ_M	is the molecular thermal conductivity of the medium.

LITERATURE CITED

1. L. A. Pigal'skaya, Kristallografiya, 14, No. 2 (1969).